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EXTENSION OF DR. STEWART'S THEOREMS TO NON-EUCLIDEAN GEOMETRY.

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[Dr. Stewart's "Theorems" have been proved geometrically by several persons. Mr. Robert Leslie Ellis, Senior Wrangler 1840, gave analytical proofs by means of a formula which is of a general interest. It is the object of this note to use this formula in the case of some closed curves and to find out the analogues of the theorems in non-euclidean space.]

1. We shall first state the analytical formula of Mr. Ellis and give its proof in the words of the author.

Lemma. If $f(\phi)$ is a rational integral function of $\sin\phi$ and $\cos\phi$, then a value may always be assigned to n, such that

$$S \equiv f(\phi) + f\left(\phi + \frac{2\pi}{n}\right) + \dots\dots + f(\phi + \frac{n-1}{n} 2\pi)$$

shall be independent of ϕ.

If $$Df(\phi) = f\left(\phi + \frac{2\pi}{n}\right),$$

the preceding expression is equivalent to

$$(1 + D + \dots D^{n-1}) f(\phi),$$

$$= \frac{D^n f(\phi) - f(\phi)}{D-1}$$

$$= \frac{1}{D-1}\{f(\phi + 2\pi) - f(\phi)\} = \Delta^{-1}(0).$$

Now obviously,

$$\Delta_{-1}(0) = \Sigma\, a_m \sin mn\,\phi + \Sigma\, b_m \cos mn\,\phi, \text{ (m integral).}$$

Hence $\quad S = \Sigma\, a_m \sin mn\,\phi + \Sigma\, b_m \cos mn\,\phi.$

Let the index of the highest power of $\sin \phi$ or $\cos \phi$ in $f(\phi)$ be p then it is easily seen that when $f(\phi)$ is developed as it may always be in a series of sines and cosines of multiples of ϕ, $(p\phi)$ will be the largest angle that can enter into the development; but if n is greater than p, $mn\phi$ will be greater than $p\phi$, except when $m = 0$. Hence the development

$$\Sigma a_m \sin mn\,\phi + \ldots\ldots$$

cannot coincide with that obtained by summing the separate development of

$$f(\phi),\ f\left(\phi + \frac{2\pi}{n}\right), \text{ etc., unless } a_m = b_m = 0$$

in every case except when $m = 0$. Hence, as $\sin mn\,\phi = 0$ when $m = 0$, the expression will be reduced to b_0 and we shall have, when $n > p$,

$$f(\phi) + \ldots\ldots f\left(\phi + \frac{n-1}{n}\, 2\pi\right) = b_0 = \text{a constant.}$$

The constant b_0 will of course be the sum of the constant parts of the development of $f(\phi)$, $f\left(\phi + \frac{2\pi}{n}\right)$, etc., and as these are n in number, it will be n times the constant part of $f(\phi)$; but this is by Fourier's theorem equal to

$$\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, d\phi.$$

Hence

$$f(\phi) + f\left(\phi + \frac{2\pi}{n}\right) + \ldots f\left(\phi + \frac{n-1}{n}\, 2\pi\right) = \frac{n}{2\pi}\int_0^{2\pi} f(\phi)\, d\phi,$$

when $n > p$, which is the analytical formula of Mr. Ellis.

2. We shall now give some of its applications to simple curves before stating Dr. Stewart's theorems and finding their analogues in *Elliptic Space*.

Ex. (1). If any n radii vectores be drawn from the centre of an ellipse at equal angles to one another, the sum of the squares of their reciprocals is equal to n times the square of the reciprocal of that radius vector which is equally inclined to the major and minor axes.

[For, we have

$$\frac{1}{r^2} = \frac{1}{b^2}(1 - e^2 \cos^2 \theta).$$

$$\therefore \qquad \sum\frac{1}{r^2} = \frac{n}{b^2} - \frac{e^2}{b^2}\sum\cos^2 \theta ;$$

and $$\sum\cos^2 \theta = \frac{n}{2\pi}\int_0^{2\pi} \cos^2 \theta \, d\theta = \frac{n}{2}, \text{ by § 1.}$$

$$\therefore \qquad \sum\frac{1}{r^2} = \frac{n}{b^2} - \frac{ne^2}{2b^2} = \frac{n}{b^2}\left(1 - \frac{e^2}{2}\right) = \frac{n}{b^2}\left(1 - e^2 \cos^2 \frac{\pi}{4}\right),$$

which proves the result.

Ex. (2). In the previous example, consider the fourth powers of the reciprocals of the radii vectores :

$$\frac{1}{r^4} = \frac{1}{b^4}(1 - e^2 \cos^2 \theta)^2 = \frac{1}{b^4}\{ 1 - 2e^2 \cos^2 \theta + e^4 \cos^4 \theta \}.$$

$$\therefore \quad \sum\frac{1}{r^4} = \frac{n}{b^4} - 2\frac{e^2}{b^4}\sum\cos^2 \theta + \frac{e^4}{b^4}\sum\cos^4 \theta.$$

Now $$\sum\cos^4 \theta = \frac{n}{2\pi}\int_0^{2\pi} \cos^4 \theta \, d\theta = \frac{3n}{8}.$$

$$\therefore \quad \sum\frac{1}{r^4} = \frac{n}{b^4} - \frac{2e^2}{b^4}\cdot\frac{n}{2} + \frac{e^4}{b^4}\cdot\frac{3n}{8} = \frac{n}{b^4}\left(1 - e^2 + \frac{3e^4}{8}\right).$$

In general, $$\sum\frac{1}{r^{2m}} = \frac{1}{b^{2m}}(1 - e^2 \cos^2 \theta)^m$$

$$= \frac{1}{b^{2m}}\Sigma[1 - {}^mC_1 e^2 \cos^2 \theta + {}^mC_2 e^4 \cos^4 \theta - \ldots\ldots \pm e^{2m} \cos^{2m} \theta]$$

$$= \frac{n}{b^{2m}} - {}^mC_1 \frac{e^2}{b^{2m}}\sum\cos^2 \theta + \ldots\ldots \pm \frac{e^{2m}}{b^{2m}}\sum\cos^{2m} \theta,$$

where $$\sum\cos^{2m} \theta = \frac{n}{2\pi}\int_0^{2\pi} \cos^{2m} \theta \, d\theta$$

$$= \frac{n}{2\pi}\left[\frac{(2m-1)\,.\,(2m-3)\ldots\ldots 1}{m\,.\,(m-1)\ldots\ldots 1}\cdot 2\pi\right].$$

Similar properties exist for the limacon and the lemniscate as may be readily seen.

Ex. (3). If n radii vectores be drawn from the centre of an ellipse equally inclined to one another to meet its inverse with respect to the centre as base, shew that $\Sigma\rho^{\frac{2}{3}}r^2$ is constant, where r is the radius vector

to the inverse curve and ρ the radius of curvature at the point on the ellipse determined by r;

$$\text{For} \quad \rho = \frac{(\text{normal})^3}{l^2}.$$

$$\therefore \quad \rho^{\frac{2}{3}} = \frac{(\text{normal})^2}{l^{\frac{4}{3}}} = (ab)^{\frac{4}{3}}\, r_1{}^2 \left(\frac{\sin^2\theta}{b^4} + \frac{\cos^2\theta}{a^4} \right),$$

where r_1 is the corresponding radius vector of the ellipse.

$$\therefore \quad \rho^{\frac{2}{3}} r^2 = (ab)^{\frac{4}{3}} k^4 \left\{ \frac{1}{b^4} - \left(\frac{1}{b^4} - \frac{1}{a^4} \right) \cos^2\theta \right\},$$

k being the radius of inversion.

$$\therefore \quad \sum \rho^{\frac{2}{3}}\, r^2 = (ab)^{\frac{4}{3}} k^4 \left\{ \frac{n}{b^4} - \frac{n}{2} \left(\frac{1}{b^4} - \frac{1}{a^4} \right) \right\}$$

which proves the result.

As significantly remarked by Mr. Ellis, every curve could have an interpretation for the analytical formula. The interpretation that we have given is applicable only to closed curves.

3. We shall now state Dr. Stewart's theorems and find out corresponding results in Elliptic Space.

The following are the theorems of Dr. Stewart :—

(1) From any point in the circumference of a circle draw perpendiculars $p, p_1, \ldots\ldots$ to the sides of a regular n-sided circumscribed polygon; then if r is the radius,

$$2\Sigma p^3 = 5nr^3. \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

Dem. Let the assumed point subtend at the centre an angle ϕ from the adjacent point of contact. Then

$$p = r(1 - \cos\phi),\ p_1 = r\left\{ 1 - \cos\left(\phi + \frac{2\pi}{n}\right) \right\}, \text{ etc.}$$

$$\therefore \quad \Sigma(p^3) = r^3\, \Sigma(1 - \cos\phi)^3.$$

By the general formula

$$\sum (1 - \cos\phi)^3 = \frac{n}{2\pi} \int_0^{2\pi} (1 - \cos\phi)^3\, d\phi$$

$$= \frac{n}{2\pi} \cdot 2^4 \int_0^{2\pi} \sin^6\theta\, d\theta \ldots\ldots (\theta = \tfrac{1}{2}\phi)$$

$$= \frac{n}{\pi} \cdot 2^3 \cdot \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \pi = \frac{5}{2^5} \cdot 2\pi.$$

$$\therefore \quad 2\Sigma p^3 = 5nr^3.$$

Cor. If the assumed point be at a distance l from the centre, instead of on the circumference, prove that

$$2\Sigma p^3 = 2nr^3 + 3nl^2 r. \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

(2) A regular n-sided polygon is inscribed in the circle and lines $c_1, c_2, \ldots\ldots$ are drawn from its corners to a point assumed on the circumference, then

$$\Sigma c^4 = 6nr^4. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Cor. If the assumed point be at a distance l from the centre, shew that

$$\Sigma c^4 = nr^4 + 4nr^2 l^2 + nl^4. \quad \ldots \quad \ldots \quad (4)$$

(3) We return to the circumscribed polygons and determine the sum of the fourth powers of the perpendiculars

$$8\Sigma p^4 = 35nr^4. \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Cor. In the general case when the assumed point is at a distance l from the centre, to shew that

$$8\Sigma p^4 = 8nr^4 + 24nr^2 l^2 + 3nl^4. \quad \ldots \quad \ldots \quad (6)$$

(4) To find now the sum of the mth powers of the perpendiculars; that is, to shew that

$$\Sigma p^m = n \cdot \frac{2m-1 \cdot 2m-3 \ldots\ldots 1}{m \cdot m-1 \ldots\ldots 1} r^m. \quad \ldots \quad (7)$$

Cor. If the assumed point is at a distance l from the centre, to shew that

$$\Sigma p^m = nr^m + n \cdot \frac{m.m-1}{1 \cdot 2} \cdot \tfrac{1}{2} r^{m-2} l^2 + n \cdot \frac{m.m-1.m-2.m-3}{4!} \cdot \frac{3.1}{4.2} r^{m-4} l^4.$$

$$+ \ldots\ldots \quad (n > m). \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

(5) Lastly, consider the $2m^{\text{th}}$ powers of the chords in the case of the inscribed polygons:

$$c^2 = 2r^2(1 - \cos\phi).$$

$$\therefore \quad \Sigma c^{2m} = 2^m r^m \, \Sigma (1 - \cos\phi)^m$$

$$= n \cdot \frac{2m-1 \cdot 2m-3 \ldots\ldots 1}{m \cdot m-1 \ldots\ldots 1} r^{2m} \ldots \quad \ldots \quad (9)$$

Cor. In the general case when the assumed point is at a distance l from the centre, we have

$$c^2 = r^2 + l^2 - 2rl\cos\phi.$$

$$\therefore \quad \Sigma c^{2m} = \Sigma(r^2 + l^2 - 2rl\cos\phi)^m$$
$$= n\,.\,(r^2 + l^2)^m + {}^mC_2\,.\,(r^2 + l^2)^{m-2}\,(2rl)^2\,\Sigma\cos^2\phi$$
$$+\ {}^mC_4\,(r^2 + l^2)^{m-4}\,(2rl)^4\,\Sigma\cos^4\phi + \ldots\ldots$$

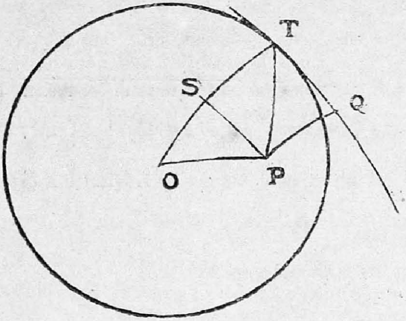
(odd powers vanish in the summation)

$$= n(r^2 + l^2)^m + {}^mC_2\,(r^2 + l^2)^{m-2}\,(2rl)\,\frac{n}{2}$$
$$+\ {}^mC_4\,(r^2 + l^2)^{m-4}\,(2rl)^4\,.\,\frac{3n}{8} + \ldots\ldots \qquad \ldots (10)$$

[Proofs of the above depend upon the following :—

$$\sum\cos^m\phi = \frac{n}{2\pi}\int_0^{2\pi}\cos^m\phi\,d\phi$$
$$= \frac{n}{2\pi}\,.\,4\,.\int_0^{\pi/2}\cos^m\phi\,d\phi \text{ (when } m \text{ is even integer)}$$
$$= 2\frac{n}{\pi}\,.\,\frac{m-1\,.\,m-3\,\ldots\ldots\,1}{2\,.\,4\,.\,6\,\ldots\ldots\,m}\,.\,\frac{\pi}{2}\Big].$$

4. To extend these propositions to non-euclidean space :



In elliptic* space, let a circle have a radius r. P, any point at a distance l from O and TQ the tangent at T and PQ ⊥ from P to the tangent and equal to p say. Draw a ⊥ PS on OT. Then the quadrilateral S′PQT is tri-rectangular ; if we write the parts

$$\left[-\frac{\pi}{2} + \angle SPQ,\ \frac{SP}{k},\ \frac{\pi}{2} - \frac{ST}{k},\ \frac{\pi}{2} - \frac{TQ}{k},\ \frac{PQ}{k}\right]$$

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* Results in hyperbolic geometry will be obtained by changing k into ik in the results of elliptic geometry and consequently need not be proved separately.

in cyclic order, then we have the rule :

sine of the middle part = product of the cosines of the opposite parts.

Hence we get

$$\sin \frac{PQ}{k} = \cos \frac{SP}{k} \sin \frac{ST}{k}$$

i.e.

$$\sin \frac{p}{k} = \cos \frac{SP}{k} \sin \frac{(r - OS)}{k}$$

$$= \cos \frac{SP}{k} \left\{ \sin \frac{r}{k} \cos \frac{OS}{k} - \cos \frac{r}{k} \sin \frac{OS}{k} \right\}$$

$$= \sin \frac{r}{k} \cos \frac{SP}{k} \cos \frac{OS}{k} - \cos \frac{r}{k} \cos \frac{SP}{k} \sin \frac{OS}{k}$$

$$= \sin \frac{r}{k} \cos \frac{l}{k} - \cos \frac{r}{k} \cos \frac{l}{k} \tan \frac{OS}{k}$$

(by solving the right-angled Δ OSP)

$$= \sin \frac{r}{k} \cos \frac{l}{k} - \cos \frac{r}{k} \cos \frac{l}{k} \tan \frac{l}{k} \cos \phi$$

$$= \sin \frac{r}{k} \cos \frac{l}{k} - \cos \frac{r}{k} \sin \frac{l}{k} \cos \phi. \quad \dots \quad \dots \quad (1)$$

In particular, if the point be on the circumference, we get

$$2 \sin \frac{p}{k} = \sin \frac{2r}{k} (1 - \cos \phi). \quad \dots \quad \dots \quad (2)$$

If the chord PT be called c, we get by solving the ΔOPT,

$$\cos \frac{c}{k} = \cos \frac{r}{k} \cos \frac{l}{k} + \sin \frac{r}{k} \sin \frac{l}{k} \cos \phi ; \quad \dots \quad (3)$$

and when P is on the circumference,

$$\cos \frac{c}{k} = \cos^2 \frac{r}{k} + \sin^2 \frac{r}{k} \cos \phi. \quad \dots \quad \dots \quad (4)$$

5. The following are the analogues of Dr. Stewart's theorems :

(1) $\sin \frac{p}{k} = \frac{1}{2} \sin \frac{2r}{k} (1 - \cos \phi)$,

$$\sin \frac{p_1}{k} = \frac{1}{2} \sin \frac{2r}{k} \left\{ 1 - \cos \left(\phi + \frac{2\pi}{r} \right) \right\}, \text{ etc.}$$

$$\therefore \sum \sin^3 \frac{p}{k} = \frac{1}{8} \sin^3 \frac{2r}{k} \sum (1 - \cos \phi)^3 = \frac{1}{8} \sin^3 \frac{2r}{k} \cdot \frac{5n}{2}$$

$$= \frac{5n}{16} \sin^3 \frac{2r}{k}.$$

Cor. $\sin\frac{p}{k} = \sin\frac{r}{k}\cos\frac{l}{k} - \cos\frac{r}{k}\sin\frac{l}{k}\cos\phi$, &c.

$$\therefore \sum \sin^3\frac{p}{k} = n\sin^3\frac{r}{k}\cos^3\frac{l}{k} - 3\sin^2\frac{r}{k}\cos^2\frac{l}{k}\cos\frac{r}{k}\sin\frac{l}{k}\sum\cos\phi$$

$$+ 3\sin\frac{r}{k}\cos\frac{l}{k}\cos^2\frac{r}{k}\sin^2\frac{l}{k}\sum\cos^2\phi - \cos^3\frac{r}{k}\sin^3\frac{l}{k}\sum\cos^3\phi$$

$$= \frac{1}{2}n\sin\frac{r}{k}\cos\frac{l}{k}\left(2\sin^2\frac{r}{k}\cos^2\frac{l}{k} + 3\cos^2\frac{r}{k}\sin^2\frac{l}{k}\right).$$

$$\therefore 2\sum\sin^3\frac{p}{k} = n\sin\frac{r}{k}\cos\frac{l}{k}\left(2\sin^2\frac{r}{k}\cos^2\frac{l}{k} + 3\cos^2\frac{r}{k}\sin^2\frac{l}{k}\right).$$

(2) $\cos\frac{c}{k} = \cos^2\frac{r}{k} + \sin^2\frac{r}{k}\cos\phi$, &c.

$$\therefore \sum\cos^2\frac{c}{k} = n\cos^4\frac{r}{k} + 2\cos^2\frac{r}{k}\sin^2\frac{r}{k}\sum\cos\phi + \sin^4\frac{r}{k}\sum\cos^2\phi$$

$$= n\cos^4\frac{r}{k} + \frac{n}{2}\sin^4\frac{r}{k}.$$

Cor. $\cos\frac{c}{k} = \cos\frac{r}{k}\cos\frac{l}{k} + \sin\frac{r}{k}\sin\frac{l}{k}\cos\phi$, &c.

$$\therefore \sum\cos^2\frac{c}{k} = n\cos^2\frac{r}{k}\cos^2\frac{l}{k} + 2\sin\frac{r}{k}\sin\frac{l}{k}\cos\frac{r}{k}\cos\frac{l}{k}\sum\cos\phi$$

$$+ \sin^2\frac{r}{k}\sin^2\frac{l}{k}\sum\cos^2\phi$$

$$= n\cos^2\frac{r}{k}\cos^2\frac{l}{k} + \frac{n}{2}\sin^2\frac{r}{k}\sin^2\frac{l}{k}.$$

(3) $\sum\sin^4\frac{p}{k} = \frac{1}{16}\sin^4\frac{2r}{k}\sum(1-\cos\phi)^4$

$$= \frac{35n}{128}\sin^4\frac{2r}{k}.$$

Cor. $\sin\frac{p}{k} = \sin\frac{r}{k}\cos\frac{l}{k} - \cos\frac{r}{k}\sin\frac{l}{k}\cos\phi$, &c.

$$\therefore \sum\sin^4\frac{p}{k} = n\sin^4\frac{r}{k}\cos^4\frac{l}{k} - 4\sin^3\frac{r}{k}\cos^3\frac{l}{k}\cos\frac{r}{k}\sin\frac{l}{k}\sum\cos\phi$$

$$+ 6\sin^2\frac{r}{k}\cos^2\frac{l}{k}\cos^2\frac{r}{k}\sin^2\frac{l}{k}\sum\cos^2\phi$$

$$- 4\sin\frac{r}{k}\cos\frac{l}{k}\sin^3\frac{l}{k}\cos^3\frac{r}{k}\sum\cos^3\phi + \cos^4\frac{r}{k}\sin^4\frac{l}{k}\sum\cos^4\phi$$

$$= n\sin^4\frac{r}{k}\cos^4\frac{l}{k} + 3n\sin^2\frac{r}{k}\cos^2\frac{l}{k}\cos^2\frac{r}{k}\sin^2\frac{l}{k} + \frac{3n}{8}\cos^4\frac{r}{k}\sin^4\frac{l}{k}.$$

(4) $\sin \frac{p}{k} = \frac{1}{2} \sin \frac{2r}{k} (1 - \cos \phi)$, &c.

$$\therefore \sum \sin^m \frac{p}{k} = \frac{1}{2^m} \sin^m \frac{2r}{k} \sum (1 - \cos \phi)^m$$

$$= \frac{1}{2^m} \sin^m \frac{2r}{k} \cdot n \cdot \frac{2m-1 \cdot 2m-3 \ldots\ldots\ldots 1}{m \cdot m-1 \ldots\ldots\ldots\ldots\ldots 1}.$$

Cor. $\sin \frac{p}{k} = \sin \frac{r}{k} \cos \frac{l}{k} - \cos \frac{r}{k} \sin \frac{l}{k} \cos \phi$, &c.

$$\therefore \sum \sin^m \frac{p}{k} = n \sin^m \frac{r}{k} \cos^m \frac{l}{k}$$

$$+ n \cdot {}^mC_2 \cdot \frac{1}{2} \sin^{m-2} \frac{r}{k} \cos^{m-2} \frac{l}{k} \cos^2 \frac{r}{k} \sin^2 \frac{l}{k}$$

$$+ \ldots\ldots \qquad \text{where } n > m.$$

(5) $\cos \frac{c}{k} = \cos^2 \frac{r}{k} + \sin^2 \frac{r}{k} \cos \phi$, &c.

$$\therefore \sum \cos^m \frac{c}{k} = n \cos^{2m} \frac{r}{k} + \frac{n}{2} {}^mC_2 \cdot \cos^{2m-4} \frac{r}{k} \sin^4 \frac{r}{k}$$

$$+ n \cdot \frac{3 \cdot 1}{4 \cdot 2} {}^mC_4 \cos^{2m-8} \frac{r}{k} \sin^8 \frac{r}{k} + \ldots$$

Cor. When the assumed point is at a distance l from the centre, we can likewise calculate $\sum \cos^m \frac{c}{k}$.

ON A CERTAIN THEOREM IN GEOMETRY.

By F. H. V. Gulasekharam, M.A.

[1. A point P in the plane of the triangle ABC is defined by the angles PAB, PBC, PCA called angular co-ordinates and denoted by α, β, γ respectively.

2. The normal co-ordinates of P and its isogonal conjugate P′ are deduced in terms of α, β, γ and α', β', γ', where $\alpha' = A - \alpha$, &c.

3. The equation to the *inconic* of ABC having P and P′ for foci is deduced in the form $\Sigma\ [(\cot\alpha + \cot\alpha')\,x]^{\frac{1}{2}} = 0.$ (areal)

(*a*) The co-ordinates of the centre of this inconic (*referred to the medial* triangle) are determined in a simple form.

(*b*) The semi-axes of the inconic are found to be

$$\rho = \frac{R \sin A \sin B \sin C}{(\Sigma \cot\alpha + \Sigma \cot\alpha') \sin\alpha \sin\beta \sin\gamma}$$

$$= \frac{R \sin A \sin B \sin C}{\cos\alpha \cos\beta \cos\gamma + \cos\alpha' \cos\beta' \cos\gamma'}$$

$$\rho'^2 = \frac{4 R_2 \sin A \sin B \sin C}{\Sigma \cot\alpha + \Sigma \cot\alpha'}.$$

(*c*) The semi latus rectum $l = 4 R \sin\alpha \sin\beta \sin\gamma$.

4. Two of Mr. V. Ramaswami Iyer's theorems and two of Mr. M. Bhimasena Rao's theorems are proved from the above results.

5. *Finally, the following new theorem is proved :—*

"*In any triangle, the radical axis of the Director circle of an inconic and the N. P. circle cuts the latter at an angle equal to the angle TOT′, where OT, OT′ are the tangents from the circum-centre O, to the inconic.*"]

1. Any point P in the plane of a triangle ABC may be defined by the angles PAB, PBC, PCA (denoted by α, β, γ respectively) which may be called the angular co-ordinates of P.

The identical relation satisfied by (α, β, γ). Let x, y, z be the *absolute* normal co-ordinates of P with respect to ABC.

Then $\dfrac{y}{z} = \dfrac{\sin(A-\alpha)}{\sin\alpha}$; $\dfrac{z}{x} = \dfrac{\sin(B-\beta)}{\sin\beta}$; $\dfrac{x}{y} = \dfrac{\sin(C-\gamma)}{\sin\gamma}$.

$$\therefore \quad \sin\alpha \sin\beta \sin\gamma = \sin(A-\alpha)\sin(B-\beta)\sin(C-\gamma) \quad \dots (1.1)$$

Writing α', β', γ' for $A-\alpha$, $B-\beta$, $C-\gamma$ respectively, the relation (1 . 1) becomes

$$\sin\alpha \sin\beta \sin\gamma = \sin\alpha' \sin\beta' \sin\gamma'. \qquad \dots \qquad \dots \quad (1\cdot 2)$$

NOTE. α', β' γ' are the angular co-ordinates of P′, the isogonal conjugate of P with respect to ABC.

2. If (α, β, γ) $(\alpha', \beta'\ \gamma')$ be the angular co-ordinates of an isogonal pair of points, the following identities are interesting and useful in what follows :—

If $S = \Sigma(\alpha)$, $S' = \Sigma'(\alpha')$, so that $S + S' = \pi$;

$\Sigma_s = \Sigma \sin 2\alpha$; $\Sigma^{s'} = \Sigma \sin 2\alpha'$;

$\Sigma_c = \Sigma \cos 2\alpha$; $\Sigma^{c'} = \Sigma \cos 2\alpha'$;

$\sigma = \Sigma \cot\alpha + \Sigma \cot\alpha'$;

then (*a*) $$\frac{\Sigma_s + \Sigma_s'}{\sin S} = \frac{\Sigma_c - \Sigma_c'}{\cos S} = 4\sigma \sin\alpha \sin\beta \sin\gamma$$
$$= 4 \cos\alpha \cos\beta \cos\gamma + 4 \cos\alpha' \cos\beta' \cos\gamma'.$$

(*b*) $$4\Sigma(\sin\alpha \sin\alpha' \sin A) + 4 \sin A \sin B \sin C$$
$$= \Sigma_s + \Sigma_s''.$$

To prove (*a*) : From (1 . 2),

$$4 \sin\alpha \sin\beta \sin\gamma = 4 \sin\alpha' \sin\beta' \sin\gamma' ;$$

and $$\sin S = \sin S' ;$$

$$\therefore \Sigma \sin(\beta + \gamma - \alpha) = \Sigma' \sin'(\beta' + \gamma' - \alpha').$$

$$\therefore \Sigma \sin(S - 2\alpha) = \Sigma' \sin(S' - 2\alpha').$$

Hence remembering that $\sin S = \sin S'$; and $\cos S = -\cos S'$, we have

$$\frac{\Sigma_s + \Sigma_s'}{\sin S} = \frac{\Sigma - \Sigma_c'}{\cos S} = k \text{ (say)}. \qquad \dots \qquad \dots \quad (2 . 1)$$

Again

$$4\sigma \sin\alpha \sin\beta \sin\gamma = 4 \sin\alpha \sin\beta \sin\gamma\, \Sigma \cot\alpha$$
$$+ 4 \sin\alpha' \sin\beta' \sin\gamma'\, \Sigma \cot\alpha'$$
$$= 4\Sigma \cos\alpha \sin\beta \sin\gamma$$
$$+ 4\Sigma' \cos\alpha' \sin\beta' \sin\gamma'.$$

Hence remembering that $\cos(\alpha + \beta + \gamma) = -\cos(\alpha' + \beta' + \gamma')$,

$$4\sigma \sin\alpha \sin\beta \sin\gamma$$
$$= 4 \cos\alpha \cos\beta \cos\gamma + 4 \cos\alpha' \cos\beta' \cos\gamma' \dots (2.2).$$
$$= \Sigma \cos(\beta + \gamma - \alpha) + \Sigma' \cos(\beta' + \gamma' - \alpha')$$
$$= \Sigma \cos(S - 2\alpha) + \Sigma' \cos(S - 2\alpha')$$
$$= \cos S\,[\Sigma_c - \Sigma_c'] + \sin S\,[\Sigma_s + \Sigma_s'].$$
$$= k \text{ (from 2 . 1)} \qquad \dots \qquad \dots \qquad \dots \quad (2 . 3)$$

Hence the results (a).

To prove (b).

$$4 \sin \alpha \sin \alpha' \sin A = \sin 2\alpha + \sin 2\alpha' - \sin 2(\alpha + \alpha')$$

$$\therefore \quad 4 \Sigma (\sin \alpha \sin \alpha' \sin A) = \Sigma_s + \Sigma_s' - \Sigma \sin 2A.$$

$$\therefore \quad 4 \Sigma (\sin \alpha . \sin \alpha' . \sin A) + 4 \sin A \sin B \sin C = \Sigma_s + \Sigma_s' . \quad \ldots \quad \ldots (2.4).$$

3. *To find the absolute normal co-ordinates of P and P′ (isogonal pairs) in terms of their angular co-ordinates.*

As before let (α, β, γ), (α', β', γ') be the angular co-ordinates; and (x_1, y_1, z_1), (x_2, y_2, z_2,) the *absolute* normal co-ordinates of P and P′ respectively. So that

$$x_1 x_2 = y_1 y_2 = z_1 z_2 = \rho'^2, \text{ say.}$$

Let a, b, c be the sides of the triangle of reference.

$$\left.\begin{aligned} &\text{Now} & a x_1 &= \rho'^2 . \frac{a}{x_2} = \rho'^2 (\cot \beta' + \cot \gamma) \\ &\text{Similarly} & b y_1 &= \rho'^2 . \frac{b}{y_2} = \rho'^2 (\cot \gamma' + \cot \alpha), \\ &\text{and} & c z_1 &= \rho'^2 . \frac{c}{z_2} = \rho'^2 (\cot \alpha' + \cot \beta). \end{aligned}\right\} \quad \ldots (3.1).$$

Hence from (3 . 1), if M denotes the area of the triangle ABC,

$$2 M = a x_1 + b y^1 + c z_1 = \rho'^2 . \sigma.$$

$$\therefore \quad \rho'^2 = \frac{2 M}{\sigma}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2).$$

Hence from (3 . 1) and (3 . 2), the absolute normal co-ordinates x_1, y_1, z_1 are determined in terms of α, β, γ.

Similarly

$$\left.\begin{aligned} a x_2 &= \rho'^2 (\cot \beta + \cot \gamma') \\ b y_2 &= \rho'^2 (\cot \gamma + \cot \alpha') \\ c z_2 &= \rho'^2 (\cot \alpha + \cot \beta'). \end{aligned}\right\} \quad \ldots \quad \ldots (3.3).$$

NOTE.—*The quantity ρ' of this section expresses the length of the semi-minor axis of the inconic of ABC having P and P′ for its foci.*

4. *To determine the normal co-ordinates of S, the middle point of P P′.*

From (3 . 1) and (3 . 3), the absolute normal co-ordinates of S are immediately obtained.

Referred to the medial triangle of ABC, the relative normal co-ordinates of S are

$$(\operatorname{cosec} \alpha \operatorname{cosec} \alpha', \quad \operatorname{cosec} \beta \operatorname{cosec} \beta', \quad \operatorname{cosec} \gamma \operatorname{cosec} \gamma'),$$

or

$$\left(\frac{\sin \beta \sin \gamma}{\sin \alpha'}, \quad \frac{\sin \gamma \sin \alpha}{\sin \beta'}, \quad \frac{\sin \alpha \sin \beta}{\sin \gamma'}\right). \quad \ldots (4)$$

5. *The inconic of ABC having P and P′ for its foci.*

(*a*) Its *centre* is the point S of § 4.

(*b*) From the co-ordinates of the centre, the *areal equation to the conic referred to ABC is*

$$\Sigma\, [\cot \alpha + \cot \alpha')\, x\,]^{\frac{1}{2}} = 0. \ldots \quad \ldots (5 . 1)$$

(*c*) *The lengths of the axes of the inconic:*

From (5 . 1), the tangential areal equation of the inconic is

$$(\cot \alpha + \cot \alpha')\, mn + (\cot \beta + \cot \beta')\, nl$$
$$+ (\cot \gamma + \cot \gamma')\, lm = 0.$$

Hence, if ρ^2, ρ'^2 be the squares of the semi-axes of the conic, Prof. Wilkinson's equation (*vide* Q. 1038. *J. I. M. S.*) immediately gives

$$\rho^2 \rho'^2 = \frac{4\, R^4 \sin^3 A \sin^3 B \sin^3 C}{\sigma^3 \sin^2 \alpha \sin^2 \beta \sin^2 \gamma}.$$

But from § 3. Note, $\rho'^2 = \dfrac{4\, R^2 \sin A \sin B \sin C}{\sigma}$. ... (5 . 2).

$$\therefore \quad \rho = \frac{R \sin A \sin B \sin C}{\sigma \sin \alpha \sin \beta \sin \gamma}. \ldots \quad \ldots (5 . 3)$$

(*d*) *The length l of the semi-latus-rectum :*

$$l = \rho'^2 / \rho = 4\, R \sin \alpha \sin \beta \sin \gamma. \quad \ldots (5 . 4)$$

6. *The equation to the line PP′.*

We have $2. \triangle\, APP' = AP.\, AP' \sin(\alpha - \alpha')$

$$= \rho'^2 . \frac{\sin(\alpha - \alpha')}{\sin \alpha \sin \alpha'} \quad \ldots \quad \ldots (6 . 1)$$

Hence, if p, q, r be the lengths of the perpendiculars from A, B, C respectively on PP′,

$$p : q : r = \frac{\sin(\alpha - \alpha')}{\sin \alpha \sin \alpha'} : \frac{\sin(\beta - \beta')}{\sin \beta \sin \beta} : \frac{\sin(\gamma - \gamma')}{\sin \gamma \sin \gamma'}.$$

Hence the *trilinear* equation to PP′, is

$$\sum \sin A . \frac{\sin(\alpha - \alpha')}{\sin \alpha \sin \alpha'} x = 0. \qquad \dots \quad (6 . 2)$$

7. *The pedal circle of the point P. w. r. t. the triangle ABC.*

Since the pedal circle is the auxiliary circle of the conic of § 5, its centre S and radius ρ are known from § 5.

The *absolute normal co-ordinates of S referred to the medial triangle are easily proved to be*

$$\frac{\rho \sin \beta \sin \gamma}{\sin \alpha'}, \quad \frac{\rho \sin \gamma \sin \alpha}{\sin \beta'}, \quad \frac{\rho \sin \alpha \sin \beta}{\sin \gamma'}. \qquad \dots \quad (7 . 1)$$

Hence the *pedal circle cuts the sides of the medial triangle of ABC at angles whose cosines are respectively*

$$\frac{\sin \beta \sin \gamma}{\sin \alpha'}, \quad \frac{\sin \gamma \sin \alpha}{\sin \beta'}, \quad \frac{\sin \alpha \sin \beta}{\sin \gamma'}. \qquad \dots \quad (7 . 2)$$

8. *To prove Mr. V. Ramaswami Iyer's Theorem* : "*The pedal circle of P cuts the N. P.* circle *at an angle equal to* $\frac{\pi}{2} - (\alpha + \beta + \gamma)$."

J. I. M. S., Q. 863.

By a well-known result, if d be the distance of a point (whose *absolute* normal co-ordinates are x, y, z) from the circum-centre of the triangle of reference, then

$$(R^2 - d^2) \sin A \sin B \sin C = \sum yz \sin A.$$

Hence from (7 . 1), if D be the distance of P from the N. P. centre,

$$(R^2/4 - D^2) \sin A \sin B \sin C = \rho^2 \sum \sin \alpha \sin \alpha' \sin A.$$

Again if θ be the angle of intersection required,

$$R \rho \cos \theta = \rho^2 + R^2/4 - D^2.$$

$$\therefore \ 4 R \rho \cos \theta . \sin A \sin B \sin C = 4\rho^2 \sin A \sin B \sin C + 4 \rho^2 \sum \sin \alpha \sin \alpha' \sin A.$$

$$= \rho^2 . [\Sigma_s + \Sigma_s'] \text{ from } (2 . 4).$$

$$\therefore \quad \cos \theta = \frac{\Sigma_s + \Sigma_s'}{4 \sigma \sin \alpha \sin \beta \sin \gamma}$$

$$= \sin(\alpha + \beta + \gamma) \text{ from § 2. } (a)$$

$$\therefore \quad \theta = \tfrac{1}{2}\pi - (\alpha + \beta + \gamma).$$

9. *To prove Mr. M. Bhimasena Rao's result*: "*The area of the triangle OPP' is Rρ cos* (α + β + γ), *where O is the circum-centre of ABC*" (J. I. M. S., Vol. IX., p. 168).

From (6 . 2), $\dfrac{\Delta\, OPP'}{\Delta\, APP'} = \dfrac{R \sum \cos A \sin A. (\cot \alpha' - \cot \alpha)}{\dfrac{2M}{a} . \sin A (\cot \alpha' - \cot \alpha)}$.

$$\therefore \Delta\, OPP' = \frac{R^2}{\sigma} \sum \frac{\sin A \cos A. \sin (\alpha - \alpha')}{\sin \alpha. \sin \alpha'}$$

from (6 . 1)

$$= \frac{R^2}{4\,\sigma \sin \alpha \sin \beta \sin \gamma} . V,$$

where $V = \sum [\cos A \sin A (4 \cos \alpha' \sin \beta' \sin \gamma' - 4 \cos \alpha \sin \beta \sin \gamma)]$

$$= \sum [\cos A \sin A (\cos \overline{S - 2\alpha} - \cos \overline{S' - 2\alpha'}$$
$$- \cos \overline{S - 2\beta} + \cos \overline{S' - 2\beta'}$$
$$- \cos \overline{S - 2\gamma} + \cos \overline{S' - 2\gamma'} + \cos S')];$$

whereas in § 2, $S = \alpha + \beta + \gamma$; $S' = \alpha' + \beta' + \gamma'$; $S + S' = \pi$.

Hence V can be thrown into the form

$$V = 4 \sin A \sin B \sin C. \cos S + U \cos S + W \sin S,$$

where $U = \sum [(\sin A \cos A - \sin B \cos B - \sin C \cos C) (\cos 2\alpha + \cos 2\alpha')]$

$$= - 2 \cos A \cos B \cos C. \sum 2 \sin A \cos (\alpha - \alpha')$$
$$= - 2 \cos A \cos B \cos C (\Sigma_s + \Sigma'_s)$$
$$= - 2 k \cos A \cos B \cos C. \sin S, \text{ from } (2 .1)$$

and $W = \sum [(\sin A \cos A - \sin B \cos B - \sin C \cos C) (\sin 2\alpha - \sin 2\alpha')]$

$$= 2 \cos A \cos B \cos C \sum 2 \sin A \sin (\alpha' - \alpha)$$
$$= 2 \cos A \cos B \cos C (\Sigma_c - \Sigma_c')$$
$$= 2 k \cos A \cos B \cos C. \cos S, \text{ from } (2 . 1)$$

Hence $V = 4 \sin A \sin B \sin C. \cos S$.

$$\therefore \Delta\, OPP' = \frac{R^2. \sin A \sin B \sin C}{\sigma. \sin \alpha \sin \beta \sin \gamma} . \cos S$$
$$= R\, \rho \cos (\alpha + \beta + \gamma).$$

Note:— (i) It can be deduced from (6 . 1) that

$$\Delta\, APP' = 2\, R\rho\, (\cos \alpha' \sin \beta' \sin \gamma' - \cos \alpha \cos \beta \sin \gamma)$$

(ii) Hence $\sin 2A. \Delta APP' + \sin 2B. \Delta BPP' + \sin 2C. \Delta CPP' = (\Sigma \sin 2A). \Delta\, OPP.$

(iii) *In fact, if Q be any point whose absolute areal coordinates are* (x, y, z) ; *and P, P′ any two points* (*not necessarily isogonal*)

$$x . \triangle APP' + y . \triangle BPP' + z . \triangle CPP' = \triangle QPP'.$$

(iv) If G be the centroid of ABC,

$$\triangle APP' + \triangle BPP' + \triangle CPP' = 3 \triangle GPP'.$$

10. *To prove* $(R^2 - OP^2)(R^2 - OP'^2) = 4R^2\rho'^2$.

By the result cited in § 8,

$$(R^2 - OP^2) \sin A \sin B \sin C = \Sigma y_1 z_1 \sin A,$$

and
$$(R^2 - OP'^2) \sin A \sin B \sin C = \Sigma y_2 z_2 \sin A.$$

$$\therefore (R^2 - OP^2)(R^2 - OP'^2) \sin^2 A \sin^2 B \sin^2 C$$
$$= (\Sigma y_1 z_1 \sin A)(\Sigma y_2 z_2 \sin A).$$
$$= x_1 x_2\, y_1 y_2\, z_1 z_2 \left(\sum \frac{\sin A}{x_1}\right)\left(\sum \frac{\sin A}{x_2}\right)$$
$$= \rho'^2 \left(\sum \rho'^2 \frac{\sin A}{x_1}\right)\left(\sum \rho'^2 \frac{\sin A}{x^2}\right)$$
$$= \rho'^2 (\Sigma x_2 \sin A)(\Sigma x_1 \sin A).$$

Hence
$$(R^2 - OP^2)(R^2 - OP'^2) = 4R^2 \rho'^2.$$

11. *To prove* (*a*) $OP . OP' = 2R . d$, where d is the distance of the N. P. centre from the centre of the pedal circle

—*Mr. V. Ramaswami Iyer.*

(*b*) *The radical axis of the N. P. circle and the pedal circle cuts the former at the angle POP′*

—*Mr. M. Bhimasena Rao.*

Let $OP . OP' = 2\lambda R$, where λ is to be hereafter determined.

Now *in the triangle OPP′*,

$$PP'^2 = 4(\rho^2 - \rho'^2).$$

From § 10, $(R^2 - OP^2)(R^2 - OP'^2) = 4 R^2 \rho'^2$

$$\therefore \quad R^2 - OP^2 - OP'^2 + 4\lambda^2 = 4\rho'^2 \qquad \ldots \quad (11 \cdot 1)$$

$$\therefore \quad R^2 + 4\lambda^2 - 4\rho^2 = OP^2 + OP'^2 - 4(\rho^2 - \rho'^2)$$
$$= OP^2 + OP'^2 - PP'^2$$
$$= 2 . OP . OP' \cos P\hat{O}P'.$$

$$\therefore R^2 + 4\lambda^2 - 4\lambda R \cos P\hat{O}P' = 4\rho^2. \qquad \ldots \qquad \ldots \quad (11 \cdot 2)$$

Again from § 9, $\Delta OPP' = R\rho \cos(\alpha + \beta + \gamma)$

$\therefore \qquad OP \,.\, OP' \sin P\hat{O}P' = 2R\rho \cos(\alpha + \beta + \gamma).$

$\therefore \qquad \lambda \sin P\hat{O}P' = \rho \cos(\alpha + \beta + \gamma).$ ... (11·3)

Again if N be the N. P. centre, and W the point of intersection of the N. P. circle and the pedal circle of P, then *in the triangle* ·NWS,

$NW = R/2\,;\ WS = \rho\,;\ NS = d\,;\ N\hat{W}S = \frac{\pi}{2} - (\alpha + \beta + \gamma)$ by § 8,

Hence $\qquad R^2 + 4d^2 - 4dR \cos S\hat{N}W = 4\rho^2,$ ... (11·4)

and $\qquad d \sin S\hat{N}W = \rho \cos(\alpha + \beta + \gamma).$ ... (11·5)

Hence from (11·2), (11·3), (11·4), (11·5),

$$\lambda = d, \quad OP \,.\, OP' = 2Rd,$$

and
$$P\hat{O}P' = S\hat{N}W.$$

12. *To prove*: " *In any triangle, the radical axis of the Director circle of any inscribed conic and the nine-point circle cuts the latter at an angle equal to the angle TOT', where OT, OT' are the tangents from the circum-centre O to the inconic.*"

Proof. From Salmon: *Conic Sections, p. 212*, we have

$$2OP \,.\, OP' \cos TOT' = OP^2 + OP'^2 - 4\rho^2.$$

Now remembering that $OP \,.\, OP' = 2Rd$,

and $\qquad R^2 + 4d^2 = OP^2 + OP'^2 + 4\rho'^2,$ (*vide* § 11),

we have $\qquad R^2 + 4d^2 - 4Rd \cos T\hat{O}T' = 4(\rho^2 + \rho'^2).$ ... (12.1)

Again, if the Director circle cuts the N. P. circle at V, in the triangle NVS, $NS = d$, $NV = R/2$, $SV^2 = \rho^2 + \rho'^2$.

$\therefore \qquad R^2 + 4d^2 - 4Rd \cos S\hat{N}V = \rho^2 + \rho'^2.$ ... (12.2)

Hence from (12.1) and (12.2) we obtain

$$S\hat{N}V = T\hat{O}T'.$$

Corollary: (i) *If the in-conic passes through the circum-centre O, the Director circle touches the N. P. circle.*
This is a well-known theorem.

Corollary: (ii) *If the Director-circle of any in-conic passes through the circum-centre, then it passes through the ends of a diameter of the N. P. circle*; and *conversely*,

if the Director-circle of any in-conic passes through the ends of a diameter of the N. P. circle, then it passes through the circum-centre as well.

[The result stated in § 12 may be considered as an extension of corollary (1) which is a particular case of the following theorem of Mr. V. Ramaswami Iyer published in the Edinburgh Mathematical Society's *Proceedings*, 1896 :—

If a conic is inscribed in a triangle and one of its confocals passes through the circum-centre, then the mutual orthoptic circle of the two conics touches the nine-points circle of the triangle.

The following extension corresponding to this theorem may be stated :—

The radical axis of the mutual orthoptic circle of a conic inscribed in a triangle and a confocal, and the nine-points circle of the triangle, cuts the latter at an angle equal to that subtended by the confocal at the circum-centre of the triangle.

Geometrical proofs of the above extension and Mr. Gulasekharam's Theorem of § 12 easily follow from the result (stated in § 11 (*b*) of this article) by the application of Question 192 of Mr. V. Ramaswami Aiyar, or Question 253. M. B.]

MATHEMATICS IN INDIA—THEN AND NOW.

BY V. SANKARAN, MANGALORE.

It is a characteristic of the present times in India that we feel as if we have been shut out from all knowledge of the glories of this ancient land in the palmy days of yore. The cry is now "Look back into your past history and you will learn more than what you can by merely studying what you are taught in schools." In every department of knowledge this looking back into the past is necessary. We have begun to realize how advanced our fore-fathers were in the study of the Humanities, Political and Social Sciences, Arts and Philosophy. It is therefore, but natural to expect that in other departments of knowledge also, they should have reached a tolerably high standard of advancement and efficiency.

The task of making available to the world at large the ancient Mss. concerning Mathematics does not seem so far to have been taken up in right earnest.

We are indeed thankful to persons like the late Prof. M. Rangacharya and Mr. Trivedi, who spared no pains to bring out Gevernment editions of *Ganita Sára Sangraha* and *Rékhá Ganita*. At present we want more men like these to make known to the people the existence of ancient mathematical treatises and their contents.

The place of the 'Preface' in modern treatises was taken up in our old works by the 'introductory verses' Thus, in the first of the above-mentioned works, viz., *Ganita Sara Sangraha*, Mahávíracharya says :—

"In all those transactions which relate to worldly, Vedic or (other) similarly religious affairs, calculation is of use. In the science of love, in the science of wealth, in music and in the drama, in the art of cooking, and similarly in medicine and in things like the knowledge of architecture.

In prosody, in poetics and poetry, in logic and grammar and such other things and in relation to all that constitutes the peculiar value of the arts—in all these, the science of computation is held in high esteem.

In relation to the movements of the sun and other heavenly bodies, in connection with eclipses and the conjunction of planets, and in connection with *triprasna* and the course of the moon—indeed in all these mathematics is utilised.

In short, whatever there is in all the three worlds, possessed of moving and non-moving beings, cannot exist as apart from measurement."

It is thus clear that mathematics was studied in those days because, it was found indispensable in ordinary life and in its study the practical aspect was always kept in view. No branch of mathematics was studied which did not have its application in life at the time in which it was written. Further, things were studied only to that degree of accuracy which was consistent with the practical needs of the people. People in those days were not interested in calculating the value of π to 707 places. For, they realized that decimal fractions to more places than 3 or 4 are useless for all practical purposes. That calculations were not carried beyond 2 or 3 places of decimals can be easily understood from a perusal of the contents of some of the mathematical treatises in Tamil and Sanskrit.

There seem to have been about a dozen mathematical works in Tamil and Sanskrit, of which Kari Nayanar says he intends writing a summary. Among mathematical books in Sanskrit, the author mentions *Anjanam*, *Daivam*, *Bhuvanadípam* and *Ganitaratnam*. Among the treatises in Tamil are mentioned *Govindanarpadi*, *Kanakkadhikáram Orambam*, *Kalambakam Tribhuvanatilakam*, *Kanakaratnam* and *Chirukanakku*. Of these, only the '*Kanakkadhikaram*' is now available and very little is known about the others. No mention is made, however, of *Lílaváti* or *Ganita Sára Sangraha*, thus showing that the '*Kanakkadhikaram* belongs to an earlier age.

Bhaskaracharya, the author of the *Lilavati*, says towards the end of his *Bijaganitam*: ब्रह्माह्वयश्रीधरपद्मनाभबीजानि यस्मादतिविस्तृतानि suggesting that there were many authors before him who had written treatises on mathematics—chief among them being Brahma Gupta, Sridhara and Padmanabha. Of these Brahma Gupta alone seems to have gained recognition, the works of the others being almost unknown. Mr. Sudhakara Dwivedi in his '*Ganakatarangini*' gives a big list of mathematicians. But as it is written in Sanskrit, its contents are not widely known.

Confining our attention to *Lilavati* and *Kanakkadikaram*, we may begin by noticing that these are works on 'general mathematics'—the analogue of what we now call 'elementary mathematics.'

It may be useful to compare them with modern treatises as regards the general treatment of the subject.

We know that most of the modern treatises contain a good many examples after each type. The type is explained and worked out and a number of questions are given on the same or mainly the same model, though with some verbal changes and with some chapters on what are

known as 'miscellaneous examples.' But in the *Lilavati* as also in the *Kanakkadikaram*, we find only one or two examples after each type. The reason for this is stated by Bhaskara towards the end of his *Bijaganita*, where he says :

नह्युदाहरणान्तोऽस्ति स्तोकमुक्तं इदं यतः ॥

* * *

प्राज्ञे शास्त्रं स्वयं याति विस्तारं वस्तुशक्तितः ॥

"Examples can be easily multiplied by the intelligent student. Hence only a few examples are given."

Again, the idea of suggesting miscellaneous problems towards the end of a treatise is not solely a modern one. Though the *Lilavati* lacks it, the same author has suggested several problems in the last chapter of the *Siddanta Siromani*, his masterpiece on Astronomy.

A second feature of the modern text-book is that it does not give ready made rules for types, while the *Lilavati* and *Kanakkadikaram* give them. The reason is not far to seek. The people in the early ages used to compress most of their learning into short mnemonic verses or sutras. Those were not the days of elaborate treatises written *in extenso*. The art of printing had not then been invented. The teacher was expected in each case to arrive at the method by independent work and in the end to give the result in the form of a rule; then, read the corresponding sloka or stanza and finally elicit from the student himself its explanation. Again, a last point with regard to the treatises of the Middle Ages is that they are almost always written in verse thus throwing some side-light on the literary advancement of the times.

Now proceeding to the contents, we find that the *Lilavati* of Bhaskara and *Ganita Sara Sangraha* of Mahavira are essentially treatises on elementary mathematics as it is understood to-day. Till very recently, the several departments of mathematical knowledge, arithmetic, algebra and geometry were regarded as watertight compartments. But now, the angle of vision has changed and instead of regarding them as separate and mutually exclusive subjects, we have begun to see in them common elements which give them a corporate and inter-dependent existence. A question is first solved arithmetically, generalisations are made therefrom, the results are then expressed algebraically and, whenever possible, illustrated geometrically. Though to most of us this might appear as an idea of recent date, we have ample grounds to believe that the notion of the inter-dependence of these subjects was in existence and has been applied in India from the earliest times. Even

to-day, if we go to any old gentlemen without any experience of western education, it will be interesting to know how he finds out the square of any number. The process by which he works out the result will suggest to any thoughtful man the application of what is known as the distributive law for multiplication. For example, to find out the square of 27, the working will be as follows: "20 times 20, 400; 20 times 7, 140; 540; 20 times 7, 140; 680; 7 times 7, 49; 729—so that the square of 27 is 729." Again, taking up another instance, the multiplication of mixed fractions, say $3\frac{1}{2}$, $4\frac{1}{2}$, the process is more patent. "3 times 4, 12; 3 times $\frac{1}{2}$, $1\frac{1}{2}$, $13\frac{1}{2}$; 4 times $\frac{1}{2}$, 2, $15\frac{1}{2}$; $\frac{1}{2}$ of $\frac{1}{2}$, $\frac{1}{4}$, $15\frac{3}{4}$." In working questions by this method, we see how unconsciously the distributive law is applied. In the "*Lilavati*," the author does not introduce the word "bija" anywhere in his book, though free use is made of the rules of algebra. For example, in the 9th sloka he says,

समद्विघातः कृतिरुच्यतेथ स्थाप्योऽन्त्यवर्गोगुणान्त्यनिघ्नः ॥

* * * *

खण्डद्वयस्याभिहतिर्द्विनिघ्निता तत्खण्डवर्गैक्य युताकृतिर्वा ॥

"The square of a quantity is obtained by multiplying it by itself. To find this, to the square of the last digit, add twice the product of this and the preceding number; and to all this add the square of the preceding; and proceed like this"—evidently applying the general formula

$$(a + b + c)^2 = a^2 + 2a\,(b + c) + (b + c)^2.$$

Another method is: "divide the number into two parts and to twice the product of these add the squares of the parts."

Again in the 13th sloka, the author says:

समत्रिघातश्चघनः प्रदिष्टः स्थाप्योघनोन्त्यस्य ततोऽन्त्यवर्गः

आदित्रिनिघ्नः तत आदिवर्गः त्र्यन्त्याहतो थादिघनश्चसर्वे ॥

* * * *

खण्डाभ्यां वाहगे राशिः त्रिघ्नः खण्डघनैक्ययुक् ॥

In the first part the author applies the expansion $a^3 + 3a^2 b + 3ab^2 + b^3$ and in the second $a^3 + b^3 + 3ab.(a+b)$ for the cube of the binomial $(a+b)$. Thus we find that the rules and formulæ of algebra are freely applied in the work.

Next, taking up the contents of the *Lilavati* in greater detail, the author begins with "abhinnaparikarmáshtakam," comprising the four rules applied to integers and raising them to the powers 2, 3, $\frac{1}{2}$ and $\frac{1}{3}$. Then comes "játichatushtayam," or conversion of mixed fractions into vulgar

fractions reducing them to equivalent fractions with common numerator or denominator and comparing them, followed by "bhinna" and "sunya pari. karmashtakam"—the eight operations aforesaid on fractions and zero. Here it will be interesting to note that the result of dividing a number by zero is left as undetermined and indeterminate being simply called "khahasa." Then follow the chapters on problems leading to simple and quadratic equations, "trairasika" and "pancharasika" (proportion, simple and compound, direct and inverse), "bhandapratibhanda" or barter, "misravyavahara" or alligations or mixtures. Then the author takes up "sredi"—simple arithmetical and geometrical progressions, the sum of the squares and cubes of the first *n* natural numbers. This is followed by "kshétra vyavahára"—mensuration of the triangle, quadrilateral, circle, sphere, rectangular parallelpiped, frusta on rectangular bases, and square and cubic measure. The last two chapters are devoted to "kuttaka" or continued fractions, and "ankapasa" or permutations and combinations. The above give in brief the contents of the *Lilavati* and we can see how far these compare with the elementary mathematics of to-day. Before we take up the other works for consideration, one thing that could be said in favour of the *Lilavati* is that it meets almost all the requirements of every-day life in India. Each one of the sections above enumerated has got an application in life. The questions set forth are mostly practical and no question is *dry*. Each has got a bearing, historical, social or religious. Thus while suggesting questions for mixtures, the author takes up the mixing up and melting together of different qualities of gold in varying proportions. A question on quadratics runs thus :

पार्थः कर्णवधाय मार्गणगणं क्रुद्धो रणे संदधे
तस्यार्धेन निवार्य तच्छरगणं मूलैश्चतुर्भिर्हयान् ।
शल्यं षडभिरथेषुभिस्त्रिभिरपि छत्रं ध्वजं कार्मुकं
चिच्छेदास्य शिरः शरेण कतिते यानर्जुनः संदधे ॥

Again a question on proportion takes the form :

प्राप्नोति चेत् षोडशवत्सरा स्त्री द्वात्रिंशतं विंशतिवत्सरा किम् ।

"If a maid-servant 16 years old gets Rs. 32 what will a maid-servant of 20 get?" Here some western historian of mathematics takes this verse to mean : "If the cost of a maid of 16 be Rs. 32, what will be the cost of a maid of 20?" and goes on to infer the social condition of the period and says that slave trade was prevalent in India in those times. The meaning itself is far-fetched and the inference unwarranted.

A question on permutations reads:

पाशांकुशाहिडमरुककपालशूलैः खट्वाङ्गशि शरचापयुतैर्भवन्ति ।
अन्योन्यहस्तकलितैः कतिमूर्तिभेदाः शम्भोर्हरेरिव गदारिसरोजशंखैः ॥

"How many different idols of Siva and Vishnu can be made by interchange of àyudhas, it being given that Siva has ten and Vishnu four different àyudhas."

These questions are essentially Indian and these appeal to the Indian reader much better than the dry and unpractical examples given in most English text-books.

Thus in point of contents, modern elementary mathematics includes the same portions as are dealt with in the *Lilavati* with something more of formal geometry and algebra.

(*To be continued.*)

SHORT NOTES.

Transformation of $(x^p - 1)/(x - 1)$.

In 'Nature' of June 9, 1921, Prof. G. B. Matthews announced the polynomials y and z in x in connection with the well-known transformation of the polynomial

$$\frac{4(x^p - 1)}{x - 1} \text{ in the form } y^2 - (-1)^{\frac{1}{2}(p-1)} pz^2,$$

(p being an odd prime number) for the value $p = 37$. Sometime later, Prof. Pocklington, F. R. S., announced the polynomials y and z for the primes 41, 43, 47, 53, 59 and 61, following the method given on pp. 216, 217 in Art. 194 of Matthew's *Theory of Numbers*, Part I.

The following is perhaps a simpler and readier method of arriving successively at the co-efficients of the powers of x in y and z and does not involve any high knowledge of the theory of numbers.

1. We may write $X = \frac{x^p - 1}{x - 1}$, ($p$ being an odd prime)

$$y = 2x^{\frac{1}{2}(p-1)} + P_1 x^{\frac{1}{2}(p-3)} + P_2 x^{\frac{1}{2}(p-5)} + \dots\dots,$$

and $$z = Q_1 x^{\frac{1}{2}(p-3)} + Q_2 x^{\frac{1}{2}(p-5)} + \dots\dots\dots$$

Now, following the notation of Prof. Matthews in his *Theory of Numbers*, let us put

$$\text{(i)} \quad P_n = -\frac{1}{n}(S_n + p_1 S_{n-1} + p_2 S_{n-2} + \dots\dots + p_{n-1} S_1)$$

where $p_r = a_r + b_r \eta_0$, η_0 and η_1 being the roots of the equation

$$\text{(ii)} \qquad \eta^2 + \eta + \frac{1 - (-1)^{\frac{1}{2}(p-1)} p}{4} = 0;$$

$S_h = \eta_0$ or η_1 according as h is a residue of p or is a non-residue of p that is, $h R p$ or $h N p$.

$$\text{Also (iii)} \quad X_1 = x^{\frac{p-1}{2}} + p_1 x^{\frac{p-3}{2}} + p_2 x^{\frac{p-5}{2}} + \dots\dots + p_{\frac{p-3}{2}} x + p_{\frac{p-1}{2}}$$

$$\equiv U + \eta_0 V,$$

where U and V are polynomials in x with integral co-efficients.

(iv) $y = 2U - V\,;\; z = -V$

2. From (iii) and (iv), we infer

(v) $y = 2\,[X_1]_{\eta_0} = -\frac{1}{2}$

Equating the co-efficients of powers of 'x' in (iv), we deduce

$$\text{(vi)} \quad \begin{cases} P_n = 2\,[p_n]_{\eta_0} = -\frac{1}{2} \\ Q_n = -\,b_n) \end{cases}$$

Evidently $\qquad [S_h]_{\eta_0} = -\frac{1}{2} = -\frac{1}{2},$

whether $\qquad h\,R\,p$, or $h\,N\,p$, since $\eta_0 + \eta_1 = -1$

When $(n-r)\,Rp$, $\quad p_r\,S_{n-r} = (a_r + b_r\,\eta_0)\,\eta_0$

$$= (a_r - b_r)\,\eta_0 - \frac{b_r}{4}\left\{1 - (-1)^{\frac{1}{2}(p-1)}\,p\right\}$$

When $(n-r)\,Np$, $\quad p_r\,S_{n-r} = (a_r + b_r\,\eta_0)\,(-1-\eta_0)$

$$= -\,a_r\eta_0 - a_r + \frac{b_r}{4}\left\{1 - (-1)^{\frac{1}{2}(p-1)}\,p\right\}$$

$$\therefore\; [p_r\,S_{n-r}]_{\eta_0} = -\tfrac{1}{2} = -\tfrac{1}{2}\,a_r + \frac{b_r}{4}\left\{1 \pm (-1)^{\frac{1}{2}(p-1)}\,p\right\},$$

the + sign being taken when $(n-r)\;Rp$ and the $-$ sign when $(n-r)\,Np$

$$\text{(vii)} \qquad = -\tfrac{1}{2}\,[p_r]_{\eta_0} = -\tfrac{1}{2} - \frac{(-1)^{\frac{1}{2}(p-1)}\,p}{4}\; e_{n-r}\;(+b_r)$$

$$= -\frac{1}{4}\,P_r - \frac{(-1)^{\frac{1}{2}(p-1)}\,p}{4}\,Q_r\,e_{n-r} \qquad \text{from (vi)}$$

where $e_{n-r} = \{(n-r) \mid p\}$, with the usual notation.

Hence, we get the reduction formula

$$P_n = -\frac{2}{n}\left(-\frac{1}{2} - \frac{1}{4}\sum_{1}^{n-1} P_r - \frac{(-1)^{\frac{1}{2}(p-1)}\,p}{4}\sum_{1}^{n-1} Q\;e_{n-r}\right)$$

$$= \frac{1}{2n}\left\{\sum_{0}^{n-1} P_r + (-1)^{\frac{1}{2}(p-1)}\,p\,Q_r\,e_{n-r}\right\}$$

(viii) $$= \frac{1}{2n} \sum_{0}^{n-1} \left\{ P_r + (-1)^{\frac{1}{2}(p-1)} pQ_r e_{n-r} \right\}, \text{ assuming } P_0 = 2.$$

3. Again, since $y + z = 2(U - V) = 2 [X_1]_{\eta_0 = -1}$

equating co-efficients of like powers of x on both sides, we get

(ix) $$P_n + Q_n = 2 [p_n]_{\eta_0 = -1}$$

When $(n-r)$ Rp,

$$[p_r S_{n-r}]_{\eta_0 = -1} = -(a_r - b_r) - b_r \frac{1 - (-1)^{\frac{1}{2}(p-1)} p}{4}$$

$$= - [p_r]_{\eta_0 = -1} + Q_r \frac{1 - (-1)^{\frac{1}{2}(p-1)} p}{4}$$

$$= - \frac{1}{2}(P_r + Q_r) + Q_r e_{r1-r} \frac{1 - (-1)^{\frac{1}{2}(p-1)} p}{4}$$

When $(n - r)$ Np,

$$[p_r S_{n-r}] = \frac{b_r}{4} (1 - (-1)^{\frac{1}{2}(p-1)} p)$$

$$= + \frac{e_{n-r} Q_r}{4} (1 - (-1)^{\frac{1}{2}(p-1)} p)$$

$$\therefore \; P_n + Q_n = - \frac{2}{n} \left[S_n + \frac{1}{4} \left\{ 1 - (-1)^{\frac{1}{2}(p-1)} p \right\} \sum_{1}^{n-1} Q_r e_{n-r} \right.$$

$$\left. - \frac{1}{4} \sum_{1}^{n-1} (P_r + Q_r)(1 + e_{n-r}) \right]$$

since when $\frac{1 + e_{n-r}}{2} = 1$ or 0, according as $(n-r)$ Rp or $(n-r)$ Np.

But $[S_h]_{\eta_0 = -1} = -1$ or 0 according as $h \; R \; p$ or $h \; N \; p$.

$\therefore \; -1 - 2 [S_h]_{\eta_0 = -1} = +1$ or -1, accg. as $h \; R \; p$ or $h \; N \; p$

$= (h/p) = e_h^{\lambda}$, with the usual notation.

$$\text{(x)} \quad \therefore \; P_n + Q_n = \frac{1}{2n} \sum_{0}^{n-1} \Big\{ (P_r + Q_r) + P_r \, e_{n-r} + (-1)^{\frac{1}{2}(p-1)} \, p \, Q_r \, e_{n-r} \Big\}$$

since $S_n = -\dfrac{1+e_n}{2}$, $P_0 = 2$, and $Q_0 = 0$.

4. Subtracting (viii) from (x), we get

$$\text{(xi)} \qquad Q_n = \frac{1}{2n} \sum_{0}^{n-1} \{ Q_r + P_r \, e_{n-r} \}.$$

Thus, we get two convenient reduction formulæ for P_n and Q_n, viz.

$$P_n = \frac{1}{2n} \sum_{0}^{n-1} \Big\{ P_r + (-1)^{\frac{1}{2}(p-1)} \, p . \, Q_r \, e_{n-r} \Big\}, \qquad \ldots \quad \text{I}$$

and $$Q_n = \frac{1}{2n} \sum_{0}^{n-1} \Big\{ Q_r + P_r \, e_{n-r} \Big\}. \qquad \ldots \qquad \ldots \quad \text{II}$$

5. From the formulæ I and II, we can easily deduce the following two formulæ which are more suitable for numerical work :

$$2n \, P_n - (2n-1) \, P_{n-1} = (-1)^{\frac{1}{2}(p-1)} \, p \, \{ Q_{n-1} - 2 \Sigma \, Q_{n-r} + 2 \Sigma \, Q_{n-r-1} \}$$

and $$2n \, Q_n - (2n-1) \, Q_{n-1} = P_{n-1} + 2 \, \{ e_n - e^{n-1} - \Sigma \, P_{n-r} + \Sigma \, P_{n-r-1} \}$$

the summation on the left-hand side extending over all the values of $r < n$ and non-residues of p.

Since the mth co-efficients of y and z from the beginning and the end are always equal numerically, it will be sufficient if we calculate the first $\dfrac{p+3}{4}$ co-efficients in y and the first $\dfrac{p-1}{4}$ co-efficients in z, when $p = 1$ (mod. 4); and the first $\dfrac{p+1}{4}$ co-efficients in y and the first $\dfrac{p-3}{4}$ co-efficients in z, when $p \equiv 3$ (mod. 4).

6. The following table gives the values of P_n and Q_n for the primes 67, 71, 73, and 79.

p.	P	Q
67	2, 1, — 16, 9, 33, — 44, — 18, 79, — 39 — 48, 75, — 35, — 14, 69, — 89, 10 106	1, 0, — 3, 3, 4, — 8, 1, 9 — 8, — 1, 7, — 8, 5, 5, — 14, 8
71	2, 1, — 17, — 26, — 5, 31, 58, 64, 60 33, — 29, — 89, — 106, — 91, — 70, — 44 — 16, — 3	1, 1, — 2, — 5, — 5, — 2, 2 6, 11, 13, 9, 2, — 3, — 6, — 8, — 8, — 7
73	2, 1, 19, 28, 61, 106, 158, 251, 322, 442, 544, 652, 783, 868, 983, 1050, 1113, 1164, 1156	1, 1, 4, 7, 12, 20, 27, 40 50, 64, 78, 89, 104, 113, 124, 131, 134, 138
79	2, 1, — 19, — 29, 24, 69, 4, — 64, 19 75, — 67, — 125, 81, 166, — 60, — 137 87, 110, — 152, — 148	1, 1, — 3, — 6, 1, 8, 0, — 7 5, 11, — 9, — 17, 8, 18, — 7 — 13, 14, 14, — 18.

A. A. Krishnaswami Aiyangar.

ASTRONOMICAL NOTES.

Interference Method in Astronomy.

1. The recent announcement of the successful application of this method at Mount Wilson, to the measurement of the angular diameters of some of the brightest stars has created such a widespread interest that a brief account of the method may not be out of place in the astronomical page of this journal. Following so closely, as it does, the theoretical researches of Eddington[1], Wilsing[2], and other investigators, it forms one of the many remarkable instances in the history of Astronomy in which theory and observation have advanced side by side, the one helping and checking the other.

2. The method itself is not in any way a recent discovery; in fact the possibilities of the method were indicated by Fizeau so early as 1868 and in 1874 Stephan made at the Marseilles Observatory some sttempts to measure the angular diameters of some of the brightest stars; these, however, were not quite successful on account of the small aperture of the telescope used.

3. The principles involved in the method may be summarised thus:—

In an astronomical telescope the image of a star (a point source of light) formed in the focal plane, is not also a point, but consists generally of a small circular bright disc in the centre surrounded by a series of diffraction rings—the angular diameter of the first ring calculated from theory is $1.22 \frac{\lambda}{D}$, where λ is the mean wave length of light and D the aperture of the telescope. If a star, which is a close double, is observed, the images will appear just separated if the angular distance between them is such that one image falls in the first diffraction ring of the other. Lord Rayleigh has thus deduced an expression $5''/D$ for the resolving power of a telescope, where D is the aperture of the object glass in inches, *i.e.*, a telescope D'' in aperture will just be able to separate the two components of a double star whose singular distance apart is $5''/D$.

4. If a narrow slit were placed in front of the object glass, the diffraction rings will be transformed into a series of fringes parallel to the edges of the slit. Michelson and later on M. Hamy have investigated the theory of the phenomena which occur when two narrow slits parallel to each other are placed in front of the object glass. It may be interesting to recount some of the results obtained by Michelson (*Phil. Mag.* 1890).

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1 Presidential Address: Section A, British Association, 1920.

2 Publications of the Potsdam Observatory, No. 76.

(1) If the object observed is a bright circular disc, the diffraction fringes will vanish for the distances of the slits given by $l = (1{\cdot}22,\ 2{\cdot}24\ 3.26\ \ldots)\ \lambda/\alpha$, where λ is the mean wave length of light and α the angular diameter of the object.

(2) In the case of a double star with components of the same brightness close to each other, if the slits are placed in such a way that their length is perpendicular to the line joining the centres of the component stars, the fringes will vanish for the distances

$$l = \left(\frac{1}{2},\ \frac{3}{2},\ \ldots\ldots\ldots\ldots\ldots\right)\frac{\lambda}{\alpha},$$

where α is the angular distance between the component stars.

5. These two results have been used with remakable success in measuring the angular distances between some very close double stars and the angular diameters of some of the brightest giant stars. The 100-inch telescope at Mount Wilson was specially directed towards this research and some very good measures have been obtained. Capella was the star first chosen for trial. Campbell and Newall had discovered its binary character by spectroscopic methods, but hitherto the largest telescope could not separate the components. By the interference method it was found possible to obtain satisfactory measures of the distance and position angle which are in sensible accord with the results obtained by spectroscopic methods.

6. Michelson then extended the method by using an interferomete) for the measurement of stellar diameters. The giant red stars with a low density and considerable luminosity were found most suitable for the purpose. The stars whose diameters have been so far measured with the 100-inch telescope at Mount Wilson are α Orionis (Betelguese)[1] and Scorpii (Antares). The angular diameter of the former was found to be 0″.047, and as its parallax has been determined as 0″.018, its linear diameter should be about 240,000,000 miles, with an uncertainty of only about 10 per cent. The results for Antares indicate that this star is of even larger linear dimensions. The angular diameter measured by the interference method is 0″.040, but as its parallax is exceedingly small, about 0″.009, its linear diameter proves to be about 400×10^6 miles. The method seems to be of immense possibilities and it may confidently be expected that further devolopments will be made, so that angular distances down to 0″.01 may be accurately measured.

T. P. Bhaskara Sastri.

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1 *Astrophysical Journal*, May 1921.

SOLUTIONS.

Question 937.

(H. R. KAPADIA) :—If n have any positive integral value except unity, and r be any positive integer which is not a perfect power, show that

$$\Sigma(n-1)/(r^n-1) = \pi^2/6\,;$$

and if $d(n)$ denote the number of divisors of n, that $\Sigma\,\{\,d(n)-1\,\}\,/r^n=1$; also that $\Sigma(n-1)/r = \Sigma 1/(r-1)^2$.

Solution by N. B. Mitra.

Consider the series $s_n = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots\ldots$ to ∞ (1), where n is any positive integer > 1.

The series is absolutely convergent and we are, therefore, at liberty to rearrange the series in any manner.

Let r be any integer which is not a perfect power ($r > 1$).

Now the series (1) can be arranged in groups like these :—

$$\left(\frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \ldots\ldots\right) + \left(\frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{27^n} + \ldots\ldots\right) + \ldots\ldots$$
$$+ \left(\frac{1}{r^n} + \frac{1}{r^{2n}} + \frac{1}{r^{3n}} + \ldots\ldots\right) + \ldots\ldots$$

where $r = 2, 3, 5, 6, 7, 10, 11\ldots$

Now the group beginning with $\frac{1}{r^n} = 1/(r^n-1)$. Hence giving to r all possible values, we get $s_n = \Sigma 1/(r^n-1)$, where Σ refers to summation with respect to r. ... ... (2)

Hence $$\sum\frac{n-1}{r^n-1} = (n-1)\,s_n.$$

Substituting 2, 3, 4 ...... in succession, for n, in this we get,

$$\sum\frac{2-1}{r^2-1} = (2-1)s_2 = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots \text{ to } \infty$$

$$\sum\frac{3-1}{r^3-1} = (3-1)s_3 = \frac{2}{2^3} + \frac{2}{3^3} + \frac{2}{4^3} + \ldots\ldots \text{ to } \infty$$

$$\sum\frac{4-1}{r^4-1} = (4-1)s_4 = \frac{3}{2^4} + \frac{3}{3^4} + \frac{3}{4^4} + \ldots\ldots \text{ to } \infty$$

and so on.

Each of these series is absolutely convergent, for s_n is so if $n > 1$; and it is easy to see that the double series on the right satisfies Cauchy's test for absolute convergence.

Adding up these, the addition on the right being performed column by column, and observing that the sum of the pth column

$$= \frac{1}{(p+1)^2}\left\{1 + \frac{2}{p+1} + \frac{3}{(p+1)^2} + \ldots\ldots \infty\right\}.$$

$$= \frac{1}{(p+1)^2}\left\{1 - \frac{1}{p+1}\right\}^{-2} = \frac{1}{p^2},$$

we get,
$$\sum_{n=2}^{n=\infty}\left\{\sum\frac{n-1}{r^n-1}\right\} = \sum_{p=1}^{p=\infty}\frac{1}{p^2} = \frac{\pi^2}{6}.$$

NOTE.—The summation on the left-side is to be performed doubly, with respect to r as well as to n; hence it would have been better to use two Σ's instead of one as printed in the question.

Let the notation $\sum\limits_{p}$ denote summation with respect to p.

(ii) It is known that $x + x^2d(2) + x^3d(3) + \ldots\ldots$ is equal to the Lambert's series $\dfrac{x}{1-x} + \dfrac{x^2}{1-x^2} + \dfrac{x^3}{1-x^3} + \ldots\ldots$ $(/x/ < 1)$.

Hence if n be as defined in the question, we have

$$x + \sum_n d(n)x^n = \frac{x}{1-x} + \sum_n \frac{x^n}{1-x^n}$$

or
$$\sum_n d(n)x^n = \frac{x^2}{1-x} + \sum_n \frac{x^n}{1-x^n}.$$

In this, put $x = 1/r^n$, where r is as defined in the question, so that $r > 1$. We get

$$\sum_n d(n)/r^n = \frac{1}{r(r-1)} + \sum_n \frac{1}{r^n-1}.$$

Also
$$\sum_n 1/r^n = 1/r^2 + 1/r^3 + \ldots\ldots = 1/r(r-1).$$

Hence
$$\sum_n \{d(n) - 1\}/r^n = \sum_n 1/(r^n - 1),$$

and $\therefore$
$$\sum_r\sum_n \{d(n) - 1\}/r^n = \sum_r\sum_n \frac{1}{r^n-1}.$$

Now the double series on the right can be easily shown to be absolutely convergent.

Hence $\sum\limits_{r}\sum\limits_{n} 1/(r^n - 1) = \sum\limits_{n}\sum\limits_{r} 1/(r^n - 1) = \sum\limits_{n} s_n$ by (2).

Now
$$\sum_n s_n = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$$
$$+ \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots\ldots$$
$$+ \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\ldots$$

The double series on the right is evidently absolutely convergent and so we shall arrive at the same result in whatever way we proceed to sum it.

Adding by columns and observing that the pth column

$$= \frac{1}{(p+1)^2} + \frac{1}{(p+1)^3} + \ldots\ldots = \frac{1}{p(p+1)},$$

we get

$$\sum_n s_n = \sum 1/p(p+1) = 1/1.2 + 1/2.3 + 1/3.4 + \ldots\ldots \text{ to } \infty = 1.$$

Hence
$$\sum_r \sum_n \{d(n) - 1\}/r^n = 1.$$

(iii) Lastly

$$1/(r-1)^2 = \frac{1}{r^2}\left(1 - \frac{1}{r}\right)^{-2} = \frac{1}{r^2} + \frac{2}{r^3} + \frac{3}{r^4} + \ldots\ldots = \sum_n (n-1)/r^n.$$

$$\therefore \qquad \sum_r \frac{1}{(r-1)^2} = \sum_r \sum_n \frac{n-1}{r^n}.$$

NOTE.—There appears to be a printing mistake in the 3rd part of the question, where on the left side of the equality r should be replaced by r^n.

Question 1011.

(SELECTED) :—In an ellipse the tangent at P cuts the directrices in Z, Z′, and the remaining tangents from Z, Z′ to the ellipse meet at T. Show that PT is normal to the ellipse and bisected by the minor axis.

Solution by N. Sundaram Aiyar, S. M. Malurkar and several others.

Let the equation to the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let TZ and TZ′ touch the ellipse at Q and R respectively. Let the eccentric angles of P, Q and R be θ, θ_1, and θ_2 respectively.

Then, the co-ordinates of T are $\left\{\dfrac{a\cos\dfrac{\theta_1+\theta_2}{2}}{\cos\dfrac{\theta_1-\theta_2}{2}}, \dfrac{b\sin\dfrac{\theta_1+\theta_2}{2}}{\cos\dfrac{\theta_1-\theta_2}{2}}\right\}$. (i)

Also $$\frac{a}{e} = \frac{a\cos\left(\dfrac{\theta+\theta_1}{2}\right)}{\cos\dfrac{\theta-\theta_1}{2}}, \quad \ldots \quad \text{(ii)}$$

and $$-\frac{a}{e} = \frac{a\cos\left(\dfrac{\theta+\theta_2}{2}\right)}{\cos\dfrac{\theta-\theta_2}{2}}, \ldots \quad \ldots \quad \text{(iii)}$$

where e is the eccentricity of the ellipse.

Putting t, t_1 and t_2, for $\tan\dfrac{\theta}{2}$, $\tan\dfrac{\theta_1}{2}$ and $\tan\dfrac{\theta_2}{2}$ respectively, we have by adding (ii) and (iii)

$$\frac{1-tt_1}{1+tt_1} + \frac{1-tt_2}{1+tt_2} = 0, \text{ or } t_1t_2 = \frac{1}{t^2}. \quad \ldots \quad \ldots \text{(iv)}$$

Again from (ii) and (iii), $-\dfrac{1}{e^2} = \dfrac{(1-tt_1)(1-tt_2)}{(1+tt_1)(1+tt_2)} = \dfrac{2-t(t_1+t_2)}{2+t(t_1+t_2)}$.

$$\therefore \quad -\frac{1+e^2}{1-e^2} = \frac{t(t_1+t_2)}{2}.$$

The co-ordinates of T now become $\left\{\dfrac{a(1-t_1t_2)}{1+t_1t_2}, \dfrac{b(t_1+t_2)}{1+t_1t_2}\right\}$,

or $$\left\{-\frac{a(1-t^2)}{1+t^2}, \frac{-2bt(1+e^2)}{(1-e^2)(1+t^2)}\right\}$$

i.e. $$\left\{-a\cos\theta, -b\sin\theta\,\frac{(2a^2-b^2)}{b^2}\right\} \ldots \quad \ldots \text{(v)}$$

Since the x — co-ordinates of P and T are respectively $a\cos\theta$ and $-a\cos\theta$, PT is bisected by the y-axis which is the minor axis.

It is now obvious that the normal at P, the equation to which is $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$, contains T, the co-ordinates of which are given in (v).

So, PT is normal to the ellipse and bisected by the minor axis.

Question 1091.

(ENQUIRER):—If a, b, c denote the radii of three circles which touch one another externally and r_1 and r_2 be the radii of the two circles which can be drawn to touch these three, prove that

$$\frac{1}{r_1} - \frac{1}{r_2} = 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Additional Solution by G. A. Srinivasan.

This has been solved in various ways in the pages of this Journal (*e.g.* Vol. XII, pp. 237-8; Vol. XIII, pp. 22-4); an additional solution, however, may not be out of place.

Let A, B, C be the centres of the given circles, a, b, c their radii; let (A) and (B) touch at P. Invert with respect to P. The inverse figure is as shown. Let p and q be the distances from P of the inverses of (A) and (B).

Then $$2a.p = 2b.q = R^2, \quad \ldots \quad \ldots \quad \text{(i)}$$
where R is the radius of inversion.

The two circles, of radii r_1 and r_2, which touch the three given circles, and the circle (C) invert into equal circles, of radius ρ say. Suppose the distances from P of the centres of these three inverse circles are in order d_1, d_2, and d.

Then, we have at once,

$$r_1 = \frac{R^2\rho}{d_1^2 - \rho^2}, \; r_2 = \frac{R^2\rho}{\rho^2 - d_2^2}, \; c = \frac{R^2\rho}{d^2 - \rho^2} \quad \ldots \quad \text{(ii)}$$

and
$$2\rho = p + q. \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

We have, from (ii),

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{d_1^2 + d_2^2 - 2\rho^2}{R^2\rho} = \frac{2d^2 + 2(2\rho)^2 - 2\rho^2}{R^2\rho}$$

(from the triangle PO_1O_2, in which O is the middle point of O_1O_2)

$$= \frac{2(d^2 - \rho^2)}{R^2\rho} + \frac{8\rho}{R^2}$$

$$= \frac{2}{c} + \frac{4(p+q)}{R^2}, \quad \text{from (ii) and (iii)}$$

$$= \frac{2}{c} + \frac{2}{a} + \frac{2}{b}, \text{ from (i).}$$

Question 1109.

(SALDHANA AND MADHAVA).—Prove that

$$e^{\lambda_1}\left[f(x)\right] = \left(1 + \frac{1}{x}\right) f(x+1)$$

where $\lambda_1 = \left(\frac{d}{dx} + \frac{1}{x}\right)$; (Bombay, B.A.)

and extend the result as follows :—

$$e^{\lambda_2}\left[\tfrac{1}{2} f(x)\right] = \left(\frac{1}{2} + \frac{1}{x} + \frac{1}{2x^2}\right) f(x+1),$$

$$e^{\lambda_3}\left[\tfrac{1}{3} f(x)\right] = \left(\frac{1}{3} + \frac{1}{x} + \frac{1}{3x^3}\right) f(x+1),$$

where $\lambda_2 = \frac{d}{dx} + \frac{2}{x}$ and $\lambda_3 = \frac{d}{dx} + \frac{3}{x}$.

Remarks by Balakram.

The results quoted in the question are (as pointed out by several gentlemen) special cases of a general theorem. This general theorem is capable of being generalized still further as follows :—

If F (k) is a function that can be expanded in ascending and descending integral powers of k, then

$$F(\Delta)\, y = x^{-r}\, F(D)\,.\,(x^r y)$$

where Δ is the operator $D + r/x = d/dx + r/x$.

Proof :—

$$\Delta y = x^{-r}\left[x^r Dy + y\,.\,r x^{r-1}\right] = x^{-r}\, D\,(x^r y),$$

$$\Delta^2 y = x^{-r}\, D\,(x^r\,.\,\Delta y) = x^{-r}\, D^2\,(x^r y),$$

$$\Delta^3 y = x^{-r}\, D^3\,(x^r y),$$

and so on, for all integral (positive) values of n.

If $\quad \Delta^{-n}\,.\,y = z,$

$$y = \Delta^n z = x^{-r}\, D^n\,(x^r z).$$

$\therefore \quad x^{-r}\, D^{-n}\,(x^r y) = z = \Delta^{-n}\, y.$

Also $\quad \Delta^0\, y = y = x^{-r}\,(x^r y) = x^{-r}\, D^0\,(x^r y).$

Hence for all integral values of n

$$\Delta^n y = x^{-r}\, D^n\,(x^r y),$$

and consequently

$$F(\Delta) y = x^{-r}\, F(D)\,.\,(x^r y)$$

Taking F to be the exponential function, we get the results given in the question.

Question 1123.

(G. S. M.) :—Find a formula for the number of regions into which space of three dimensions is divided by n planes.

Solution by N. B. Mitra, and also by the proposer.

We shall consider the most general case ; viz the case in which no wo of the planes are parallel, no three of them pass through the same straight line and no four of them have a common point. Let F (N) denote the required number. Also let $f(n)$ denote the number into which a plane is divided by n straight lines in it, no two of which are parallel and no three of which meet in a point.

Now consider the nth plane. It is cut by the other planes in $n-1$ straight lines and these $n-1$ lines divide the nth plane into $f(n-1)$ parts. And these parts are such that if any one of them were removed, two regions into which space is divided would become one.

Hence $$F(n) = F(n-1) + f(n-1). \qquad \ldots \quad (1)$$

Now to find $f(n)$. We observe that the nth straight line is cut by each of the remaining $n-1$ lines and therefore it is divided into n parts Now there are two parts of the plane on two sides of each of these parts of the nth line which would become one if the nth line were removed.

Hence $$f(n) = f(n-1) + n.$$

Putting n equal to $n-1, n-2, \ldots\ldots 2$ in succession in this and adding all the equations, we get

$$f(n) = n + (n-1) \ldots\ldots + 2 + f(1) \text{ and } f(1) = 2.$$

Thus $$f(n) = 1 + \tfrac{1}{2}n(n+1).$$

Hence from (1), $F(n) = F(n-1) + \frac{1}{2}n(n-1) + 1.$

Putting $n-1, n-2, \ldots\ldots 3, 2$ in succession for n in this and adding all the results, we get

$$F(n) = \tfrac{1}{2}\Sigma n(n-1) + n - 1 + F(1)$$
$$= (n-1)n(n+1)/6 + n + 1, \qquad \text{since } F(1) = 2$$
$$= (n^3 + 5n + 6)/6.$$

QUESTIONS FOR SOLUTION.

1195. (K. J. SANJANA, M.A.) :—From the point P (p, q, r) perpendiculars PM, PN are drawn to the planes given by the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0:$$

prove that the length of MN is given by

$$\frac{\{A^{-1}(qG-rH)^2+B^{-1}(rH-pF)^2+C^{-1}(pF-qG)^2\}^{\frac{1}{2}}}{\{A+B+C-\frac{1}{4}(a+b+c)^2\}^{\frac{1}{2}}}$$

where A, B, C, F, G, H have their usual values.

1196. (K. J. SANJANA, M.A.) :—Solve the equation

$$\frac{d^3y}{dx^3}-3nx^{n-1}\frac{d^2y}{dx^2}+\{3n^2x^{2n-2}-3n(n-1)x^{n-2}\}\frac{dy}{dx}$$
$$-\{n^3x^{3n-3}-3n^2(n-1)x^{2n-3}+n(n-1)(n-2)x^{n-3}\}y=0.$$

1197. (N. B. MITRA):—If $2n+1$ is a prime p, prove that the numerator of $2^{4n}+p\left(1+\frac{1}{2}\ \ldots\ldots+\frac{1}{n}\right)-1\equiv 0\ (\text{mod. } p^2)$.

1198. (F. H. V. GULASEKHARAM) :—The major axis of an inscribed conic of a triangle ABC is given by the trilinear equation

$$p\alpha+q\beta+r\gamma=0.$$

(i) Prove that the equation of the conic is

$$\Sigma\{p(p^2-q^2-r^2+2qr\cos A)\}^{\frac{1}{2}}=0.$$

(ii) If the axis be fixed in direction, the centre traces out a rectangular hyperbola through the in-centre and the e-centre, *w. r. t.* which the original triangle is self-polar.

1199. (F. H. V. GULASEKHARAM):—Prove that the radius of the pedal circle of a point P with respect to a triangle ABC is given by

$$\frac{R\sin A\sin B\sin C}{\cos PAB\,.\cos PBC\,.\cos PCA+\cos PAC\,.\cos PCB\,.\cos PBA},$$

where R is the circum-radius of ABC.

If the pedal triangle of P is in perspective with ABC, prove that the above expression reduces to

$$\tfrac{1}{2}R\,.\frac{\sin A\sin B\sin C}{\cos PAB\,.\cos PBC\,.\cos PCA}.$$

1200. (MARTYN M. THOMAS) :—Show that the parabola

$$(a^2y^2-b^2x^2)\sin 2\theta-2abxy\cos 2\theta+b^2x^2+a^2y^2-4a^2b^2$$
$$+2\sqrt{2ab}\ \{(bx-ay)\sin\theta+(bx+ay)\cos\theta\}=0,$$

where θ is variable, osculates its envelope.

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1201. (MARTYN M. THOMAS) :—A periodic comet when at an angular distance θ from the perihelion of its orbit, of eccentricity e, suddenly encounters a resistance which brings it to a stand-still. Show that it will fall into the sun in time $\frac{t}{4\sqrt{2}}\left(\frac{1-e^2}{1+e\cos\theta}\right)^{\frac{3}{2}}$ where t is the periodic time of the comet.

1202. (SELECTED) :—Being given that

$$x = \frac{(1+uv)(u-v)}{(1-uv)^2},\ y = \frac{2\{uv(1-u^2)(1-v^2)\}^{\frac{1}{2}}}{(1-uv)^2},$$

find what $dx^2 + dy^2$ becomes, in terms of u, v, du, dv; and explain accurately the geometrical meaning of the result.

1203. (C. N. SREENIVASA IYENGAR) :—If Q (x) be the quotient, and R (x) the remainder when $f(x)$ is divided by its $(2n+1)^{\text{th}}$ derivative, then the roots of Q (x).R $(x) = 0$ separate those of $f(x) = 0$, supposed to be all real.

1204. (A. A. KRISHNASWMI AIYANGAR) :—Conics are drawn having quadritactic contact with the Folium of Descartes at one point and bitactic contact elsewhere. Show that the chord of contact envelopes a rectangular hyperbola.

1205. (A. A. KRISHNASWAMI AIYANGAR) :—If n be an even number, prove that all the non-repeated factors of $(2^n - 1)$ are prime to n and that the repeated factors occur once oftener than in n.

1206. (V. RAMASWAMI IYER) :—Three collinear points F_1, F_2, F_3, being the foci of a confocal system of Cartesian ovals, prove (1) that there are two ovals T_1, T_2 of the system that have vertices C_1, C_2 respectively, which are points of undulation ; (2) that the vertices of equilateral triangles described on F_1F_2, F_2F_3, F_3F_1 both ways, are six of the points of intersection of T_1 and T_2 ; (3) that the centres of these equilateral triangles lie on a circle X whose centre is the mean centre of F_1, F_2, F_3 and which cuts the axis in C_1, C_2 ; (4) that the circle Y concentric with X and of double its radius cuts the axis in O_1, O_2 the triple foci of T_1, T_2 ; (5) that F_1, F_2, F_3 are the projections on the axis of the vertices of two equilateral triangles inscribable in Y ; and (6) that the two remaining intersections of T_1, T_2 are the points with respect to which F_1, F_2, F_3 invert into the vertices of equilateral triangles.

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